ASYMPTOTIC RELATION BETWEEN THE TEMPERATURE AND HEAT FLUX AT A SURFACE FOR LARGE TIMES IN PROBLEMS WITH MOVING BOUNDARIES

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Formulas are obtained that are suitable for finding the asymptotic law of boundary motion in Stefan-type problems for an arbitrary (including nonlinear) condition on the boundary mentioned without a preliminary determination of the temperature field.

Stefan-type problems on determining the front propagation velocity (reaction, phase transition front, etc.) can be reduced to solving a functional equation describing the relationship between the dependent variable (temperature, concentration), its gradient on the moving boundary, and an additional physical condition (see [1, p. 56], say). An attempt is made in this paper to find the dependence between the quantities $T_s(t)$, $q_s(t)$, and $l(t)$ for large times. We set up such a dependence for "small" and "medium" times earlier [2, 3] for the case of a semiinfinite domain. Certain functional relationships are found in [4, 5] for a finite domain under the condition of a particular kind on the moving boundary. Below we obtain these relationships by a substantially different method that does not require specification of the physical condition.

1. SEMIINFINITE DOMAIN

The heat transfer in a coordinate system coupled to the moving boundary is described by the differential equation

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial x^2} - \dot{l}(t)\frac{\partial T}{\partial x} = 0,\ 0 < x < \infty,\ 0 < t < \infty,$$
$$T|_{x=0} = T_s(t),\ T|_{x=\infty} = 0,\ T|_{t=0} = 0. \tag{1}$$

The functional relationship between the boundary temperature T_s and its gradient on the boundary $q_s = (\partial T/\partial x)_{x=0}$ was found earlier in the form of a series [2]

$$-\sqrt{a}\,q_s = \left[D^{1/2} + \frac{\dot{l}}{2\sqrt{a}} + \frac{\dot{l}^2}{8a}D^{-1/2} - \frac{\ddot{l}}{8\sqrt{a}}D^{-1} - \left(\frac{\dot{l}^4}{128a^2} + \frac{\dot{l}\,\ddot{l}}{16a}\right)D^{-3/2} + \dots\right]T_s. \tag{2}$$

A recurrent relationship governing the multipliers for the operators $D^{-n/2}$ was also presented in [2].

The fractional differentiation operators are defined by the expression

$$D^{\nu}f(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t (t-\tau)^{-\nu} f(\tau)\,d\tau,\ -\infty < \nu < 1. \tag{3}$$

For $\nu < 0$ an equivalent definition can be obtained by integration by parts

$$D^{\nu}f(t) = \frac{1}{\Gamma(-\nu)}\int_0^t (t-\tau)^{-\nu-1} f(\tau)\,d\tau,\ -\infty < \nu < 0. \tag{4}$$

We use the following properties of the fractional derivative below:

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$$D^{\nu} f(t) g(t) = \sum_{n=0}^{\infty} \binom{\nu}{n} \frac{d^n f(t)}{dt^n} D^{\nu-n} g(t), \tag{5}$$

$$D^{\nu} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)} t^{\mu-\nu}, \; \mu+\nu-1>0. \tag{6}$$

The convergence of the series (2) for sufficiently general classes of functions has not been established; however, it can be shown that if the terms of the series decrease as $t \to 0$ or $t \to \infty$ so that $a_{n+1}/a_n \to 0$, then there is at least asymptotic convergence for sufficiently small and large values of t.

We later assume that $T_s(t)$ and $l(t)$ are monotonically increasing functions of the time, the case T_s = const is also possible; $l(t)$ has all derivatives with respect to t. We show that the limiting relation of T_s and q_s is represented by two general and two special forms depending on the rate of growth of T_s and l as $t \to \infty$. We assume the surface temperature to grow not more rapidly than t^{μ}, $0 \leqslant \mu < \infty$ as $t \to \infty$; the derivatives of T_s also grow no more rapidly than the derivatives of t^{μ}.

Let us consider the general case of $l(t)t^{-1/2} \to 0$ as $t \to \infty$ (the front is propagated more slowly than $t^{1/2}$). Substituting the majorizing function $l(t) = t^{\nu}$ in (2), taking account of (6) and (4), we have

$$-\sqrt{a} q_s = \frac{\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)} t^{\mu-\frac{1}{2}} + \frac{\nu}{2\sqrt{a}} t^{\mu+\nu-1} + \frac{\nu^2}{8a} \frac{\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{3}{2}\right)} t^{\mu+2\nu-\frac{3}{2}} - \frac{\nu(\nu-1)}{8\sqrt{a}} \frac{\Gamma(\mu+1)}{\Gamma(\mu+2)} t^{\mu+\nu-1} + \ldots$$

As $t \to \infty$ the first term of the series is dominant for $0 \leq \nu < 1/2$. Therefore, the relationship between T_s and q_s has the form

$$-\sqrt{a}\, q_s(t) \approx D^{1/2} T_s(t), \; t \to \infty. \tag{7}$$

It is possible to arrive at an analogous deduction by setting $l = t^{1/2}/\ln t$.

Relationship (7) is indepndent of the velocity of front propagation and agrees in form with that for the case when the heat is propagated in a fixed medium.

If $l(t)t^{-1/2} \to \infty$ as $t \to \infty$ (the front propagates more rapidly than $t^{1/2}$), the members of the series (2) increase in absolute value with growth of the number for $t \to \infty$; however one of the components is dominent in each factor for the different operators $D^{-\nu}$. By discarding inessential terms, we arrive at the expression

$$-\sqrt{a} q_s \approx \left[\frac{\dot{l}}{2\sqrt{a}} + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{\dot{l}^2}{4a} \right)^n D^{\frac{1}{2}-n} - \frac{\ddot{l}}{8\sqrt{a}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\dot{l}^2}{4a} \right)^n D^{-1-n} \right] T_s, \; t \to \infty. \tag{8}$$

Using (4) and (5) it can be seen by direct substitution that (8) is the finite expression

$$-\sqrt{a}\, q_s \approx \frac{\dot{l}}{2\sqrt{a}} T_s + J_1 - J_2,$$

$$J_1 = \exp\left[\left(-\frac{\dot{l}^2}{4a} \right)^* t \right] D^{1/2} \exp\left[\left(\frac{\dot{l}^2}{4a} \right)^* t \right] T_s,$$

$$J_2 = \frac{\ddot{l}}{8\sqrt{a}} \int_0^t \exp\left[\left(-\frac{\dot{l}^2}{4a} \right) (t-\tau) \right] T_s(\tau)\, d\tau. \tag{9}$$

The symbol ( )* indicates that in the calculation of $D^{1/2}$ by means of (3) with the quantity $l^2/4a$ it should behave as a constant independent of t.

It can be shown that as $t \to \infty$ $J_1 \to (\dot{l}/2\sqrt{a}) T_s$, $J_2 \to (\ddot{l}\sqrt{a}/2\dot{l}^2) T_s$. It is seen that $J_2/J_1 \to 0$. Conserving the terms dominant as $t \to \infty$ in (9), we obtain the desired relationship between the temperature and its gradient at the boundary

$$-a q_s(t) \approx \dot{l}(t) T_s(t), \; t \to \infty. \tag{10}$$

The latter corresponds in shape with the stationary dependence for T_s, q_s, $\dot{l}$ = const.

For $\nu \geqslant 1$ dependence (10) can be obtained by the method elucidated in [7] in application to our problem. Multiplying the heat-conduction equation (written in a fixed coordinate system) by $\exp(-\sqrt{p}x)$ and integrating between the limits $(l(t), \infty)$ we obtain the relationship

$$e^{-pat}\int\limits_{l(t)}^{\infty} Te^{-\sqrt{p}\,x}\,dx + \int\limits_{0}^{t}\{[\dot{l}(\tau)+a\sqrt{p}\,]\,T_s(\tau)+aq_s(\tau)\}\,e^{-pa\tau-\sqrt{p}\,l(\tau)}d\tau = 0.$$

Setting p = 0 and noting that $\int\limits_{l(t)}^{\infty} Tdx \leqslant \text{const}\cdot T_s$ (this latter is easily established by integrating the initial equation for $\nu = 1$), we arrive at the deduction that the second integral cannot grow more rapidly than T_s, for which satisfaction of the relationship (10) as $t \to \infty$ is required.

In the intermediate case when the boundary moves according to a parabolic law ($l(t)t^{-1/2} \to \beta$ = const as $t \to \infty$), all the terms of the series (2) are of identical order. It is seen that the relationship between T_s and q_s can be written in terms similar to both (7) and (10)

$$-\sqrt{a}\,q_s(t) \approx C_1(\beta)\,D^{1/2}\,T_s(t) \approx C_2(\beta)\frac{\dot{l}(t)}{\sqrt{a}}\,T_s(t),\ t \to \infty. \tag{11}$$

The distinction is in the constant factor C_1 and C_2 for whose determination we have no simple methodology. Known methods of solving the problem (1) for $l = \beta t^{1/2}$ should be used [8, 9].

Now, let us examine the case when the surface temperature T_s is an "exponential growth" function, that grows for $t \to \infty$ as $\exp \alpha t$ or $t^{\mu}\exp \alpha t$, $0 \leqslant \mu < \infty$.

Let us assume that $l(t)t^{-1} \to 0$ as $t \to \infty$ (the front is propagated more slowly than t). Substituting the majorizing function $l = t^{\nu}$, $0 \leqslant \nu < 1$ in (2) and taking into account the for $t \to \infty$ $D^{\nu}\,t^{\mu}\,\exp\alpha t \to t^{\mu}\alpha^{\nu}\exp\alpha t$ we find that the first term of the series is dominant and, therefore, the asymptotic relationship (7) is satisfied. In the case when $l(t)t^{-1} \to \infty$ as $t \to \infty$ (the front is propagated more rapidly than t), an analysis of (9) results in the dependence (10).

For the intermediate case $l(t)t^{-1} \to \beta$ = const as $t \to \infty$, the problem (1) is solved easily:

$$-\sqrt{a}\,q_s \approx \left[\frac{\beta}{2\sqrt{a}} + \exp\left(-\frac{\beta^2 t}{4a}\right)D^{1/2}\exp\left(\frac{\beta^2 t}{4a}\right)\right]T_s,\ \ t\to\infty. \tag{12}$$

The expression (12) can be written in one of the forms (11).

Therefore, critical values of the front propagation velocity have been established for which the law relating T_s, q_s, and l changes form. For the critical values the form of the relation differs from that for the mentioned general cases.

2. FINITE DOMAIN

For the problem

$$\left(\frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2}\right)T = 0,\ 0 < x < l(t),\ 0 < t < \infty,$$

$$T|_{x=0} = T_s(t),\ T|_{x=l(t)} = 0,\ T|_{t=0} = 0 \tag{13}$$

after substituting $\xi = x/l(t)$, $\tau = t$, the "stopping boundary" [10], we obtain

$$\left[\frac{\partial}{\partial\tau} - \xi\,\frac{\dot{l}(\tau)}{l(\tau)}\frac{\partial}{\partial\xi} - \frac{a}{l^2(\tau)}\frac{\partial^2}{\partial\xi^2}\right]T = 0,\ 0 < \xi < 1,\ 0 < \tau < \infty. \tag{14}$$

This latter equation is later used for the analysis.

Let us turn attention to the fact that we seek limit relationships on this section only on the fixed boundary of an expanding domain (x = 0), by keeping in mind that the relationship on the moving boundary ($x = l(t)$) can be obtained analogously to (18) for "slow" motion, while for "rapid" motion it agrees with that for the semiinfinite domain.

Let us examine the case when T_S is a "power-law growth" function. If $l(t)t^{-1/2} \to 0$ as $t \to \infty$, by selecting the majorizing function in the form $l = t^{\nu}$, $0 \leqslant \nu < 1/2$, $T_S = t^{\mu}$, we arrive after the substitution [10] $z = \tau^{1-2\nu}/(1-2\nu)$ at the problem

$$\left(\frac{\partial}{\partial z} - \frac{\nu}{1-2\nu}\frac{\xi}{z}\frac{\partial}{\partial \xi} - a\frac{\partial^2}{\partial \xi^2}\right)T = 0,\ 0<\xi<1,\ 0<z<\infty,$$

$$T_s(z) = [(1-2\nu)z]^{\mu/(1-2\nu)},\quad T|_{\xi=1} = 0,\ T|_{z=0} = 0. \tag{15}$$

We see by direct substitution that the function

$$T = [(1-2\nu)z]^{\mu/(1-2\nu)}(1-\xi) \tag{16}$$

is the solution (14) for $z \to \infty$ if the inequality

$$0 \leqslant \mu < 1-2\nu \tag{17}$$

is satisfied. After going from (16) over to the initial variables, we obtain the desired relationship

$$-q_s(t) \approx T_s(t)/l(t),\ t \to \infty, \tag{18}$$

which agrees in form with the quasistationary solution.

According to (16), the temperature gradient is constant in the whole domain for $t \to \infty$ and is determined from (18). By writing the latter for $x = l(t)$, it may be able to compare with the result from [6], where it is shown that the following relationship is satisfied on the moving boundary (in our notation)

$$\sum_{n=1}^{\infty}\frac{1}{(2n)!}\frac{d^n}{dt^n}\left(\frac{l^2}{a}\right)^n = \frac{T_s(t)}{(L\rho/\lambda)a},$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=l(t)} = -(L\rho/\lambda)\dot{l},\quad T|_{x=l(x)} = 0.$$

As $t \to \infty$ only the first term is retained in the sum for $\nu < 1/2$ and by combining the formulas written down, we obtain (18), valid for any μ, exactly. Therefore, the constraint (17) is removed.

In the special case $l = \beta t$, $\beta = \text{const}$, the solution of (14) with the conditions (13) can be found by the method in [10], which results in a final expression analogous to (7). Therefore, in a domain expanding according to a linear law, the temperature gradient on the fixed boundary is the same for $t \to \infty$ as in the fixed semiinfinite domain.

In considering the general case $l(t)t^{-1/2} \to \infty$ as $t \to \infty$ we set $l(t) = t^{\nu}$, $1/2 < \nu < \infty$. Then after the substitution $z = \tau^{1-2\nu}/(1-2\nu)$ we rewrite (14) in the form

$$\left(\frac{\partial}{\partial z} + \frac{\nu}{2\nu-1}\frac{\xi}{z}\frac{\partial}{\partial \xi} - a\frac{\partial^2}{\partial \xi^2}\right)T = 0,\ 0<\xi<1,\ -\infty<z<0. \tag{19}$$

Physically it is evident that the law (7), established above for $\nu = 1$, is also satisfied for $\nu > 1$ since if the heat in a domain expanding according to a linear laws is propagated as in a semiinfinite domain as $t \to \infty$, then even more such a regularity will be satisfied for domains expanding more rapidly than t.

For $1/2 < \nu < \infty$ the numerical factor $\nu/(2\nu-1)$ on the first derivative with respect to the coordinate in (19) varies between the limits ∞ and 1/2. Since solutions of this equation are analytic functions of the parameter $\nu/(2\nu-1)$ within the mentioned limits, the relationship (7), which is valid for $1 \leqslant \nu < \infty$, is also valid for $1/2 < \nu < \infty$. In the special case $l(t)t^{-1/2} \to \beta = \text{const}$, the desired relationship can be written for $t \to \infty$ in a form analogous to (11):

$$-\sqrt{a}\,q_s(t) \approx C_3(\beta)D^{1/2}T_s(t) \approx C_4(\beta)T_s(t)/l(t),\ t \to \infty,$$

where the constants C_3 and C_4 can be determined by the known methods in [9].

If T_S is an "exponential growth" function for the case $\nu < 1/2$ as $t \to \infty$ we did not obtain the relationship between T_S, q_S, and l. Consideration of the particular case $l(t)t^{-1} \to \beta$ results in (7). For the general case $l(t)t^{-1/2} \to \infty$ as $t \to \infty$ reasoning analogous to that presented above yields the relationship (7).

Therefore, in contrast to Sec. 1, the constant velocity of front motion is not critical for the relationship on the fixed boundary. The case $l(t)t^{-1/2} \to \beta = \text{const}$ as $t \to \infty$ is investigated by known methods of [9].

NOTATION

a, thermal diffusivity; a_n, terms of the series in (2); C, constants; D^ν, fractional differentiation symbol; f, g, arbitrary functions of the time; q, temperature gradient; l, coordinate of the moving boundary; T, temperature; L, heat of the phase transition; t, time; x, coordinate; z, variable dependent on the time; α, β, constants; μ, ν, constant exponents in the power laws; ρ, density; λ, heat conduction; ξ, dimensionless coordinate; τ, variable of integration, time; s, surface $x = 0$ or $\xi = 0$; the dot denotes differentiation with respect to the argument.

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